Are Bosonic Replicas Faulty?

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Motivated by the ongoing discussion about a seeming asymmetry in the performance of fermionic and bosonic replicas, we present an exact, nonperturbative approach to *both* fermionic and bosonic zero-dimensional replica field theories belonging to the broadly interpreted $\beta = 2$ Dyson symmetry class. We then utilise the formalism developed to demonstrate that the bosonic replicas *do* correctly reproduce the microscopic spectral density in the QCD-inspired chiral Gaussian unitary ensemble. This disproves the myth that the bosonic replica field theories are intrinsically faulty.

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Introduction.—Since the mid-1990s, there has been a revived interest in the field theoretic approaches tailor-made to the analysis of *interacting* disordered and quantum chaotic systems. In particular, the exact Keldysh [1] and approximate supersymmetry [2] techniques have been conceived to offer a nonperturbative alternative to the notoriously known *replica field theories* [3, 4, 5] whose legitimacy has been questioned [6] for more than two decades. Sadly, the newly proposed field theoretic approaches [1, 2] have not yet evolved into efficient calculational tools, and their success [7] has been very limited.

At the same time, substantial progress [8, 9, 10] was achieved over the past few years in resolving controversies surrounding nonlinear replica σ models. Specifically, the *fermionic* version [4, 11] of a replica field theory considered in the so-called zero-dimensional (random-matrix-theory [12]) limit was proven [8] to be exactly integrable. This observation brought into play the whole machinery of the theory of integrable hierarchies [13] and eventually resulted in reconstructing [8, 9] the exact spectral densities and/or correlation functions for the paradigmatic Gaussian unitary ensemble (GUE), the QCD inspired chiral GUE (chGUE) [14] and Ginibre's ensemble [15] of complex non-Hermitean random matrices. The exact fermionic replicas also have immediate implications for the nonperturbative physics of the 1D lattice impenetrable bosons [16]. The supersymmetric variation [10] of exact replicas [8] has already produced important new results [17] for the QCD at nonzero chemical potential.

The present Letter, prompted by the ongoing discussion [6, 18, 19, 20] about a seeming asymmetry in the performance of fermionic and bosonic replicas [21] (fermionic-bosonic dichotomy), addresses the problem of integrability of zero-dimensional *bosonic* replica field theories much in line with the ideas of Refs. [8, 9]. Having formulated a general nonperturbative theory of *both* fermionic and bosonic replicas, we further concentrate on the chGUE matrix model and develop an integrable theory of the corresponding nonlinear *bosonic* replica σ model. Contrary to the claims made in the literature [18], the latter is shown to produce the *exact* expression for the chGUE density of eigenlevels in the physically relevant limit of infinite-dimensional matrices. This achievement, representing the main outcome of our study, provides strong evidence that *the bosonic replicas are as good and reliable as the fermionic ones.* We conjecture that the above statement holds in the whole generality, no matter what particular random matrix model is being treated.

How replicas arise and why they are tricky.—Replica field theories (be it the original bosonic formulation invented by Wegner [3] or its fermionic counterpart [4] further extended by Finkelstein [5] to accommodate the interaction effects) are based on the identity

$$\log Z = \lim_{n\to\pm 0} \frac{Z^n - 1}{n} \tag{1}$$

which can be very useful [22] in evaluating the average $\langle \log Z\rangle$. Upon assigning Z the meaning of a quantum partition function $Z(\varsigma) = \prod_{\alpha=1}^{p} \det(\varsigma_\alpha - \mathcal{H})$ of a system characterised by a stochastic Hamiltonian $\mathcal{H}$, the identity (1) can be utilised to represent the average p-point Green function $G(\varsigma) = \langle \prod_{\alpha=1}^{p} \mathrm{Tr}\,(\varsigma_\alpha - \mathcal{H})^{-1}\rangle$ in terms of the average characteristic polynomials

$$Z_n(\varsigma) = \left\langle \prod_{\alpha=1}^{p} \det{}^n(\varsigma_\alpha - \mathcal{H}) \right\rangle, \quad \varsigma = (\varsigma_1, \cdots, \varsigma_p), \tag{2}$$

to be referred to as the replica partition function. Notice that Eq. (2) is defined for $n \in \mathbb{R}$ [23]. The recipe, known as the replica limit, reads

$$G(\varsigma) = \lim_{n\to\pm 0} \frac{1}{n^p}\, \partial_{\varsigma_1} \cdots \partial_{\varsigma_p} Z_n(\varsigma). \tag{3}$$

Equation (3) assumes a mutual commutativity of the following operations: the replica limit, differentiation, disorder averaging denoted by the angular brackets $\langle \cdots \rangle$, and a thermodynamic limit, if necessary.

Seemingly innocent at first glance, the prescription (3) is much trickier than one could naïvely expect. Indeed, in order to calculate the replica partition function $Z_n(\varsigma)$ nonperturbatively, a field theorist interprets $Z^n(\varsigma)$ in Eq. (2) as a substitute for $|n| \in \mathbb{Z}^+$ identical noninteracting copies, or replicas, of the original random system.

Each copy, exemplified by the product $\prod_{\alpha=1}^{p} \det(\varsigma_\alpha - \mathcal{H})$ of $p \geq 1$ single determinants, is represented by a functional integral over an auxiliary field which is either fermionic or bosonic by nature, depending on the sign of n. Exponentiating a random Hamiltonian $\mathcal{H}$, such a representation facilitates a nonperturbative averaging over the ensemble of stochastic Hamiltonians in Eq. (2) and eventually results in effective field theories defined on either a compact [4] (fermionic, $n \in \mathbb{Z}^+$) or a noncompact [3] (bosonic, $n \in \mathbb{Z}^-$) manifold. Such a replica mapping, $Z_{n\in\mathbb{R}}(\varsigma) \stackrel{\text{map}}{\longrightarrow} \tilde{Z}_{n\in\mathbb{Z}^\pm}(\varsigma)$, clearly indicates the key problem of replicas. By derivation, the validity of $\tilde{Z}_{n\in\mathbb{Z}^\pm}(\varsigma)$ is restricted to $n \in \mathbb{Z}^\pm$, which is not enough for implementing the replica limit (3) determined by the behaviour of $\tilde{Z}_n(\varsigma)$ in the *vicinity* of $n = 0$. This mismatch between the "available" and the "needed" is at the heart of the trickery with which the replica field theories are often charged [6].

The canonical way to bridge this gap is to determine $\tilde{Z}_n(\varsigma)$ for $n \in \mathbb{Z}^\pm$, and then attempt to analytically continue $\tilde{Z}_n(\varsigma)$ away from n integers, in general, and to a proper vicinity of $n = 0$, in particular. Since performing an analytic continuation based on an *approximate* result is a mathematically questionable procedure, the evaluation of $\tilde{Z}_n(\varsigma)$ must be done *exactly*. Below, we will show how such a nonperturbative calculation can be carried out in quite a general setting. The approach to be presented applies to the matrix models belonging to the broadly interpreted Dyson's $\beta = 2$ symmetry class [12, 24] and is by far more flexible and efficient than the one of Ref. [8].

Nonperturbative approach to replicas.—Let us concentrate on the fermionic and/or bosonic replica field theories whose mapped partition functions admit the eigenvalue representation (n is supposed to be positive)

$$\tilde{Z}_n^{(\mathrm{f/b})}(\varsigma) = \int_{\mathcal{D}^n} \prod_{k=1}^{n} d\lambda_k \, \Gamma(\varsigma; \lambda_k)\, e^{-V_n(\lambda_k)} \cdot \Delta_n^2(\boldsymbol{\lambda}). \quad (4)$$

Here $V_n(\lambda)$ is a "confinement potential" which may depend on the replica index $\pm n$; $\Gamma(\varsigma; \lambda)$ is a function accommodating relevant physical parameters ς of the theory [they are not necessarily the energies specified in Eq. (2)]. To treat the fermionic and bosonic replicas on the same footing, the integration domain $\mathcal{D}$ was chosen to be [25] $\mathcal{D} = \bigcup_{j=1}^{r} [c_{2j-1}, c_{2j}]$.

To detemine the replica partition function $\tilde{Z}_n^{(\mathrm{f/b})}(\varsigma)$ nonperturbatively, we adopt the "deform-and-study" approach, a standard string theory method of revealing hidden structures. Its main idea consists of "embedding" $\tilde{Z}_n^{(\mathrm{f/b})}(\varsigma)$ into a more general theory of τ functions

$$\tau_n^{(s)}(\varsigma; \boldsymbol{t}) = \frac{1}{n!} \int_{\mathcal{D}^n} \prod_{k=1}^{n} d\lambda_k \, \Gamma(\varsigma; \lambda_k) \times e^{-V_{n-s}(\lambda_k)}\, e^{v(\boldsymbol{t};\lambda_k)} \cdot \Delta_n^2(\boldsymbol{\lambda}) \quad (5)$$

which posses the infinite-dimensional parameter space $\boldsymbol{t} = (t_1, t_2, \cdots)$ arising as the result of the $\boldsymbol{t}$-deformation $v(\boldsymbol{t}; \lambda) = \sum_{j=1}^{\infty} t_j \lambda^j$; the auxiliary parameter s is assumed to be an integer, $s \in \mathbb{Z}$. Studying the evolution of τ functions in the extended $(n, s, \boldsymbol{t}, \varsigma)$ space allows us to identify the highly nontrivial, nonlinear differential hierarchical relations between them. Miraculously, a projection of these relations, taken at $s = 0$, onto the hyperplane $\boldsymbol{t} = \boldsymbol{0}$,

$$\tilde{Z}_n^{(\mathrm{f/b})}(\varsigma) = n!\, \tau_n^{(s)}(\varsigma; \boldsymbol{t})\Big|_{\substack{\boldsymbol{t}=\boldsymbol{0} \\ s=0}}, \quad (6)$$

generates, among others, a closed nonlinear differential equation for the replica partition function $\tilde{Z}_n^{(\mathrm{f/b})}(\varsigma)$. Since this *nonperturbative* equation appears to contain the replica (or hierarchy) index n as a parameter, it is expected [8] to serve as a proper starting point for building a consistent analytic continuation of $\tilde{Z}_n^{(\mathrm{f/b})}(\varsigma)$ away from n integers.

Having formulated the crux of the method, let us turn to its detailed exposition. The two key ingredients of the exact theory of τ functions are (i) the bilinear identity [13] and (ii) the (linear) Virasoro constraints [26].

(i) The bilinear identity encodes an infinite set of hierarchically structured nonlinear differential equations in the variables $\{t_j\}$. For the model introduced in Eq. (5), the bilinear identity reads [27, 28]:

$$\oint_{\mathcal{C}_\infty} dz\, e^{a\, v(\boldsymbol{t}-\boldsymbol{t'};z)} \left(\tau_n^{(s)}(\boldsymbol{t} - [\boldsymbol{z}^{-1}])\, \frac{\tau_{m+1}^{(m+1+s-n)}(\boldsymbol{t'} + [\boldsymbol{z}^{-1}])}{z^{m+1-n}}\, e^{v(\boldsymbol{t}-\boldsymbol{t'};z)} - \tau_m^{(m+s-n)}(\boldsymbol{t'} - [\boldsymbol{z}^{-1}])\, \frac{\tau_{n+1}^{(s+1)}(\boldsymbol{t} + [\boldsymbol{z}^{-1}])}{z^{n+1-m}} \right) = 0. \quad (7)$$

Here, $a \in \mathbb{R}$ is a free parameter; the integration contour $\mathcal{C}_\infty$ encompasses the point $z = \infty$; the notation $\boldsymbol{t} \pm [\boldsymbol{z}^{-1}]$ stands for the infinite set of parameters $\{t_j \pm z^{-j}/j\}$; for brevity, the physical parameters ς were dropped from the arguments of τ functions.

Being expanded in terms of $\boldsymbol{t'} - \boldsymbol{t}$ and a, Eq. (7) generates four integrable hierarchies. One of them, the Kadomtsev-Petviashvili (KP) hierarchy in the Hirota form [29]

$$\frac{1}{2} D_1 D_k\, \tau_n^{(s)}(\boldsymbol{t}) \circ \tau_n^{(s)}(\boldsymbol{t}) = s_{k+1}([\boldsymbol{D}])\, \tau_n^{(s)}(\boldsymbol{t}) \circ \tau_n^{(s)}(\boldsymbol{t}) \quad (8)$$

($k \geq 3$) is of primary importance for the exact theory

of replicas [31]. The first nontrivial member of the KP hierarchy reads

$$\left(\partial_{t_1}^4 + 3\,\partial_{t_2}^2 - 4\,\partial_{t_1}\partial_{t_3}\right)\,\log \tau_n^{(s)}(\boldsymbol{\varsigma};\boldsymbol{t}) + 6\,\left(\partial_{t_1}^2\,\log \tau_n^{(s)}(\boldsymbol{\varsigma};\boldsymbol{t})\right)^2 = 0. \tag{9}$$

In what follows, it will be shown that its projection onto $s=0$ and $\boldsymbol{t}=\boldsymbol{0}$ [Eq. (6)] gives rise to a nonlinear differential equation for the replica partition function $\tilde{Z}_n^{(\mathrm{f/b})}(\boldsymbol{\varsigma})$.

(ii) Since we are interested in deriving a differential equation for $\tilde{Z}_n^{(\mathrm{f/b})}(\boldsymbol{\varsigma})$ in terms of the derivatives over *physical parameters* $\{\varsigma_j\}$, we have to seek an additional block of the theory that would make a link between the $\{t_j\}$ derivatives in Eq. (9) taken at $\boldsymbol{t}=\boldsymbol{0}$ and the derivatives over physical parameters $\{\varsigma_j\}$. The study [32] by Adler, Shiota, and van Moerbeke suggests that the missing block is the *Virasoro constraints* which reflect the invariance of the τ function [Eq. (5)] under a change of the integration variables. In the present context, it is useful to demand the invariance under the transformation

$$\lambda_j \to \mu_j + \epsilon\mu_j^{q+1} f(\mu_j) \prod_{k=1}^{\dim(\boldsymbol{c}')} (\mu_j - c_k'),\ \ \epsilon>0, \tag{10}$$

where $\boldsymbol{c}' = \{c_1,\cdots,c_{2r}\}\setminus\{\pm\infty\}$. The function $f(\lambda)$ is related to the confinement potential $V_{n-s}(\lambda)$ through the parameterisation $dV_{n-s}/d\lambda = -g(\lambda)/f(\lambda)$, where $g(\lambda) = \sum_{j=0}^\infty b_j\lambda^j$ and $f(\lambda)=\sum_{j=0}^\infty a_j\lambda^j$ depend on n and s.

The transformation (10) induces the Virasoro-like constraints that can be written in the additive form

$$\left[\hat{\mathcal{L}}_q^V(\boldsymbol{t}) + \hat{\mathcal{L}}_q^\Gamma(\boldsymbol{\varsigma};\boldsymbol{t})\right]\tau_n^{(s)}(\boldsymbol{\varsigma};\boldsymbol{t}) = 0,\quad q\geq -1. \tag{11}$$

The operators $\hat{\mathcal{L}}_q^V(\boldsymbol{t})$ and $\hat{\mathcal{L}}_q^\Gamma(\boldsymbol{t})$ are associated with the $e^{-V_{n-s}(\lambda)}$ and the $\Gamma(\boldsymbol{\varsigma};\lambda)$ parts of the integrand, respectively; also, $\hat{\mathcal{L}}_q^{\Gamma=1}(\boldsymbol{t})\equiv 0$. The first operator $\hat{\mathcal{L}}_q^V(\boldsymbol{t})$ can be expressed in terms of the Virasoro operators [26]

$$\hat{\mathcal{L}}_q(\boldsymbol{t}) = \sum_{j=1}^\infty jt_j\,\partial_{t_{q+j}} + \sum_{j=0}^q \partial_{t_j}\partial_{t_{q-j}},\quad q\geq -1, \tag{12}$$

which depend solely on the symmetry of the replica field theory and obey, for all $p,q\geq -1$, the Virasoro algebra $[\hat{\mathcal{L}}_p,\hat{\mathcal{L}}_q] = (p-q)\hat{\mathcal{L}}_{p+q}$. [Equation (12) assumes that ∂_{t_0} is identified with the multiplicity of the matrix integral in Eq. (5), $\partial_{t_0}\equiv n$]. Explicitly, it holds [27] that

$$\hat{\mathcal{L}}_q^V(\boldsymbol{t}) = \sum_{k=0}^{\dim(\boldsymbol{c}')} s_{\dim(\boldsymbol{c}')-k}(-[\boldsymbol{\sigma}])\sum_{\ell=0}^\infty \left(a_\ell\hat{\mathcal{L}}_{q+k+\ell}(\boldsymbol{t}) - b_\ell\partial_{t_{q+k+\ell+1}}\right),\quad [\boldsymbol{\sigma}]_j = \frac{1}{j}\sum_{k=1}^{\dim(\boldsymbol{c}')}(c_k')^j,\quad q\geq -1. \tag{13}$$

Here, $s_k(\boldsymbol{t})$ are the Schur polynomials [30].

While very similar in spirit, the calculation of $\hat{\mathcal{L}}_q^\Gamma(\boldsymbol{t})$, the second ingredient in Eq. (11), is more of an art since the function $\Gamma(\boldsymbol{\varsigma};\lambda)$ in Eq. (5) may significantly vary from one replica model to the other.

Remarkably, for $\boldsymbol{t}=\boldsymbol{0}$, the two equations [Eqs. (9) and (11)] can be solved jointly to bring a closed nonlinear differential equation for $\tilde{Z}_n^{(\mathrm{f/b})}(\boldsymbol{\varsigma})$. It is this equation which, being supplemented by appropriate boundary conditions, provides a truly nonperturbative description of the replica partition functions and facilitates performing the replica limit.

Chiral GUE and bosonic replicas.—To see the above formalism at work and also answer the question raised in the title of our Letter, let us consider the $N\times N$ chGUE random matrices

$$\mathcal{H}_{\mathcal{D}} = \begin{pmatrix} 0 & \mathcal{W} \\ \mathcal{W}^\dagger & 0\end{pmatrix} \tag{14}$$

known to describe the low-energy sector of SU($N_c\geq 3$) QCD in the fundamental representation [14]. Composed of rectangular $n_L\times n_R$ random matrices $\mathcal{W}$ with the Gaussian distributed complex-valued entries

$$P_{n_L,n_R}(\mathcal{W}) = \left(\frac{2\pi}{N\Sigma^2}\right)^{n_L n_R}\exp\left[-\frac{N\Sigma^2}{2}\mathrm{tr}\,\mathcal{W}^\dagger\mathcal{W}\right], \tag{15}$$

where $N = n_L+n_R$, the matrix $\mathcal{H}_{\mathcal{D}}$ has exactly $\nu = |n_R-n_L|$ zero eigenvalues identified with the topological charge ν; the remaining eigenvalues occur in pairs $\{\pm\lambda_j\}$; the parameter Σ denotes the chiral condensate.

To determine the (microscopic) spectral density from the bosonic replicas, we define the replica partition function $Z_n^{(\mathrm{b})}(\varsigma) = \left\langle \det^{-n}(\varsigma + i\mathcal{H}_{\mathcal{D}})\right\rangle_{\mathcal{W}}$ and map it onto a bosonic field theory. In the half-plane $\mathrm{Re}\,\varsigma>0$, the partition function $Z_n^{(\mathrm{b})}(\varsigma)$ reduces to [18, 20]

$$\tilde{Z}_n^{(\mathrm{b})}(\omega) = \int_{\mathcal{S}_n} d\mu_n(\boldsymbol{\Omega})\,\det{}^{\nu-n}\boldsymbol{\Omega}\,\exp\left[-\frac{\omega}{2}\mathrm{Tr}(\boldsymbol{\Omega}+\boldsymbol{\Omega}^{-1})\right], \tag{16}$$

where the integration domain $\mathcal{S}_n$ spans all $n\times n$ positive definite Hermitean matrices $\boldsymbol{\Omega}$. Equation (16) was derived in the thermodynamic limit $N\to\infty$ with the spectral parameter $\omega = \varsigma N\Sigma$ being kept fixed ($\mathrm{Re}\,\omega>0$).

Spotting the invariance of the integrand in Eq. (16) under the unitary rotation of the matrix $\boldsymbol{\Omega}$, one readily realises that $\tilde{Z}_n^{(\mathrm{b})}(\omega)$ belongs to the class of τ functions

specified by Eq. (5) where $\mathcal{D}$ is set to $\mathbb{R}^+$, the potential V_{n-s} is $V_{n-s}(\lambda) = (n-s-\nu)\log\lambda$, and $\Gamma(\varsigma;\lambda)$ is replaced with $\Gamma(\omega;\lambda) = \exp\left[-(\omega/2)(\lambda+\lambda^{-1})\right]$. This observation implies that the associated τ function $\tau_n^{(s)}(\omega;\boldsymbol{t})$ satisfies both the first KP equation (9) and the Virasoro constraints (11) with [27]

$$\hat{\mathcal{L}}_q^V(\boldsymbol{t}) = \hat{\mathcal{L}}_{q+1}(\boldsymbol{t}) + (\nu-n+s)\,\partial_{t_{q+1}}, \quad \hat{\mathcal{L}}_q^\Gamma(\omega;\boldsymbol{t}) = -\frac{\omega}{2}\partial_{t_{q+2}} - \delta_{q,-1}\left(\omega\partial_\omega + \frac{\omega}{2}\partial_{t_1}\right) + [1-\delta_{q,-1}]\,\frac{\omega}{2}\partial_{t_q}. \tag{17}$$

Projecting Eq. (9) taken at $s=0$ onto $\boldsymbol{t}=\boldsymbol{0}$, and expressing the partial derivatives therein via the derivatives over ω with the help of Eqs. (11) and (17), we conclude that $\tilde{Z}_n^{(\mathrm{b})}(\omega) = n!\,\tau_n^{(0)}(\omega;\boldsymbol{0})$ obeys the differential equation [27]

$$h_n''' + \frac{2}{\omega}h_n'' - \left(4 + \frac{1+4(n^2+\nu^2)}{\omega^2}\right)h_n' + 6(h_n')^2 + \frac{1-4(n^2+\nu^2)}{\omega^3}h_n - \frac{2}{\omega^2}(h_n)^2 + \frac{4}{\omega}h_n h_n' + \frac{4n^2}{\omega^2} = 0 \tag{18}$$

that can be reduced to the Painlevé III. Here $h_n(\omega) = \partial_\omega \log \tilde{Z}_n^{(\mathrm{b})}(\omega)$. Considered together with the boundary conditions $h_n(\omega\to 0) \simeq -n\nu/\omega$ and $h_n(\omega\to\infty) \simeq -n - n^2/(2\omega)$, following from Eq. (5), the nonlinear differential equation (18) provides a nonperturbative characterisation of $\tilde{Z}_n^{(\mathrm{b})}(\omega)$ for all $n\in\mathbb{Z}^+$.

To pave the way for the replica calculation of the Green function $G(\omega)$ determined by the replica limit $G(\omega) = -\lim_{n\to 0} n^{-1} h_n(\omega)$, one has to analytically continue $h_n(\omega)$ away from n integers. The previous studies [8, 9] suggest that the sought analytic continuation is given by the very same Eq. (18) where the replica parameter n is let to explore the entire real axis. This leap makes the rest of the calculation straightforward. Representing $h_n(\omega)$ in the vicinity of $n=0$ as $h_n(\omega) = \sum_{p=1}^\infty n^p a_p(\omega)$, we conclude that $G(\omega) = -a_1(\omega)$ satisfies the equation

$$\omega^3 G''' + 2\omega^2 G'' - \left(1+4\nu^2+4\omega^2\right)\omega G' + (1-4\nu^2)\,G = 0. \tag{19}$$

Its solution, subject to the boundary conditions consistent with those specified below Eq. (18), brings the microscopic spectral density $\varrho(\omega) = \pi^{-1}\mathrm{Re}\,G(i\omega+0)$ in the form

$$\varrho(\omega) = \nu\delta(\omega) + \frac{\omega}{2}\left[J_\nu^2(\omega) - J_{\nu-1}(\omega)J_{\nu+1}(\omega)\right]. \tag{20}$$

Obtained within the framework of *bosonic* replicas, this celebrated formula provides strong evidence against the idea of their inapplicability to the nonperturbative description of random matrix spectra [6], in general, and of the chGUE spectra [18], in particular. In view of the previous study [8] on the performance of *fermionic* replicas, we are led to speculate that truly nonperturbative approaches to nonlinear replica σ models leave no room for the fermionic-bosonic dichotomy.

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